Efficient Densest Subgraph Computation in Evolving Graphs

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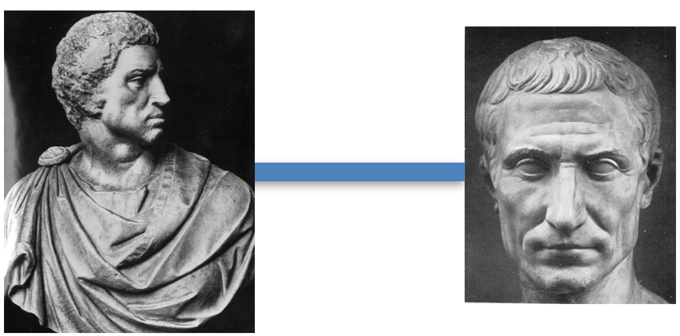
BROWN

Joint work with Silvio Lattanzi (Google Research, NY) and Mauro Sozio (Télécom ParisTech)

Social Networks are Constantly Evolving

RECENT ACTIVITY

Julius Caesar is now friends with Brutus. Like · Comment



Brutus Julius

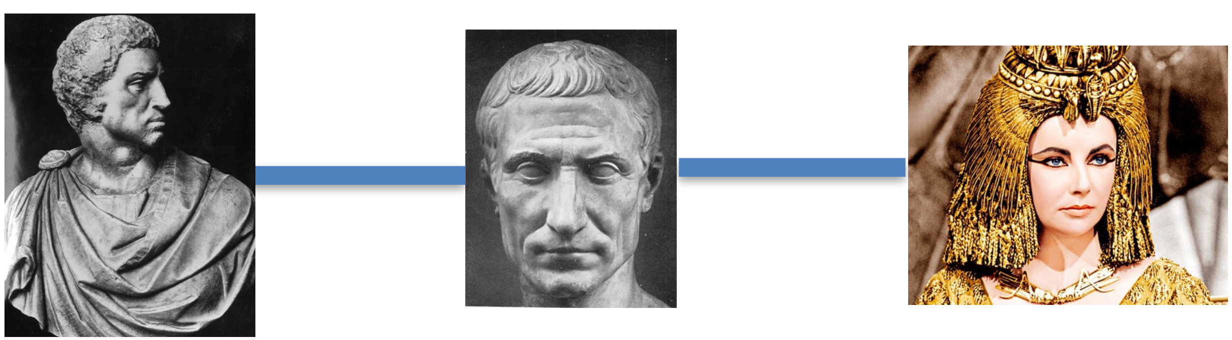
Social Networks are Constantly Evolving

Julius Caesar is in a relationship with Cleopatra.

2000 Years Ago Like · Comment

Mark Anthony likes this.

Add a comment....

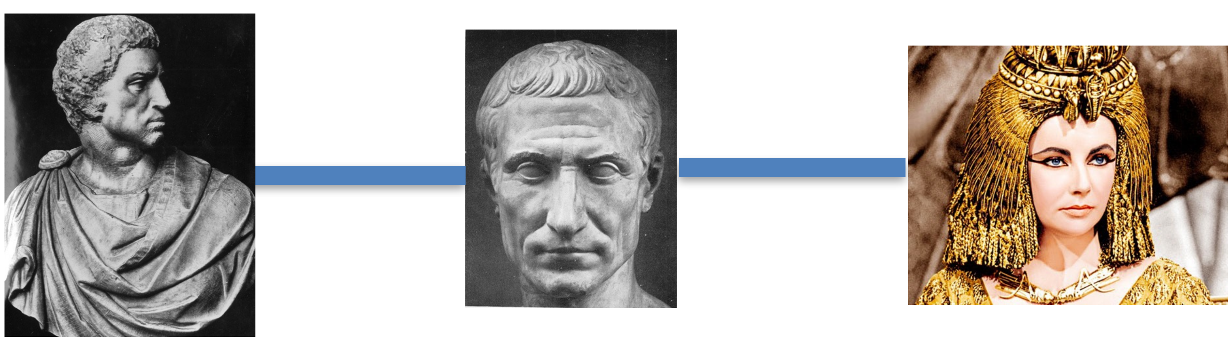


Brutus Julius Cleopatra

Social Networks are Constantly Evolving

RECENT ACTIVITY

Brutus attended Caesar's Betrayal in Senate.



Brutus Julius Cleopatra

Social Networks are Constantly Evolving

Julius Caesar
You too @Brutus???

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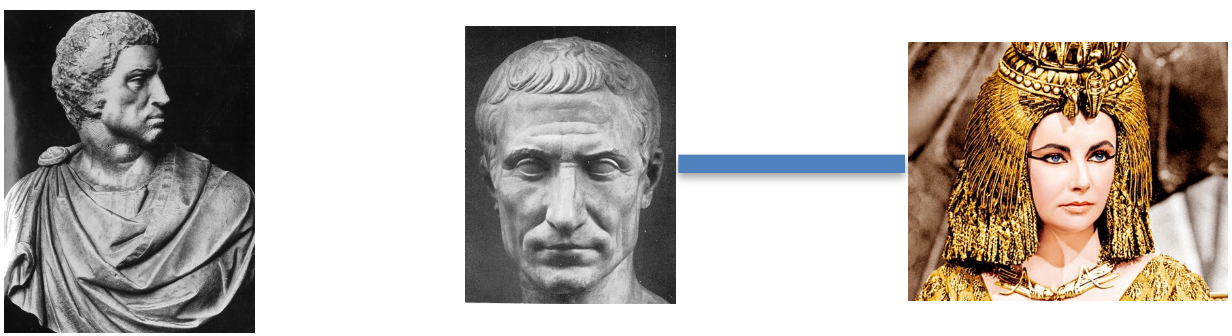

Brutus Julius Cleopatra

Social Networks are Constantly Evolving

Brutus

Cleopatra

Social Networks are Constantly Evolving

Cleopatra is in a relationship with Mark Anthony.

2000 Years Ago Like · Comment

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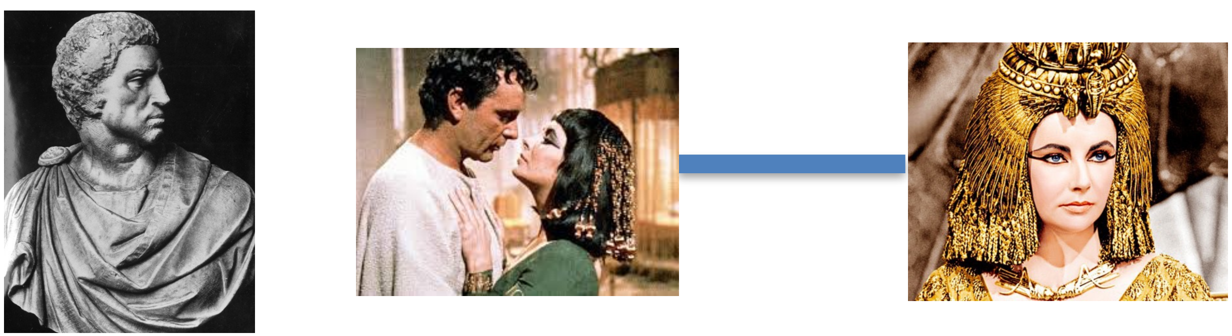
Add a comment....

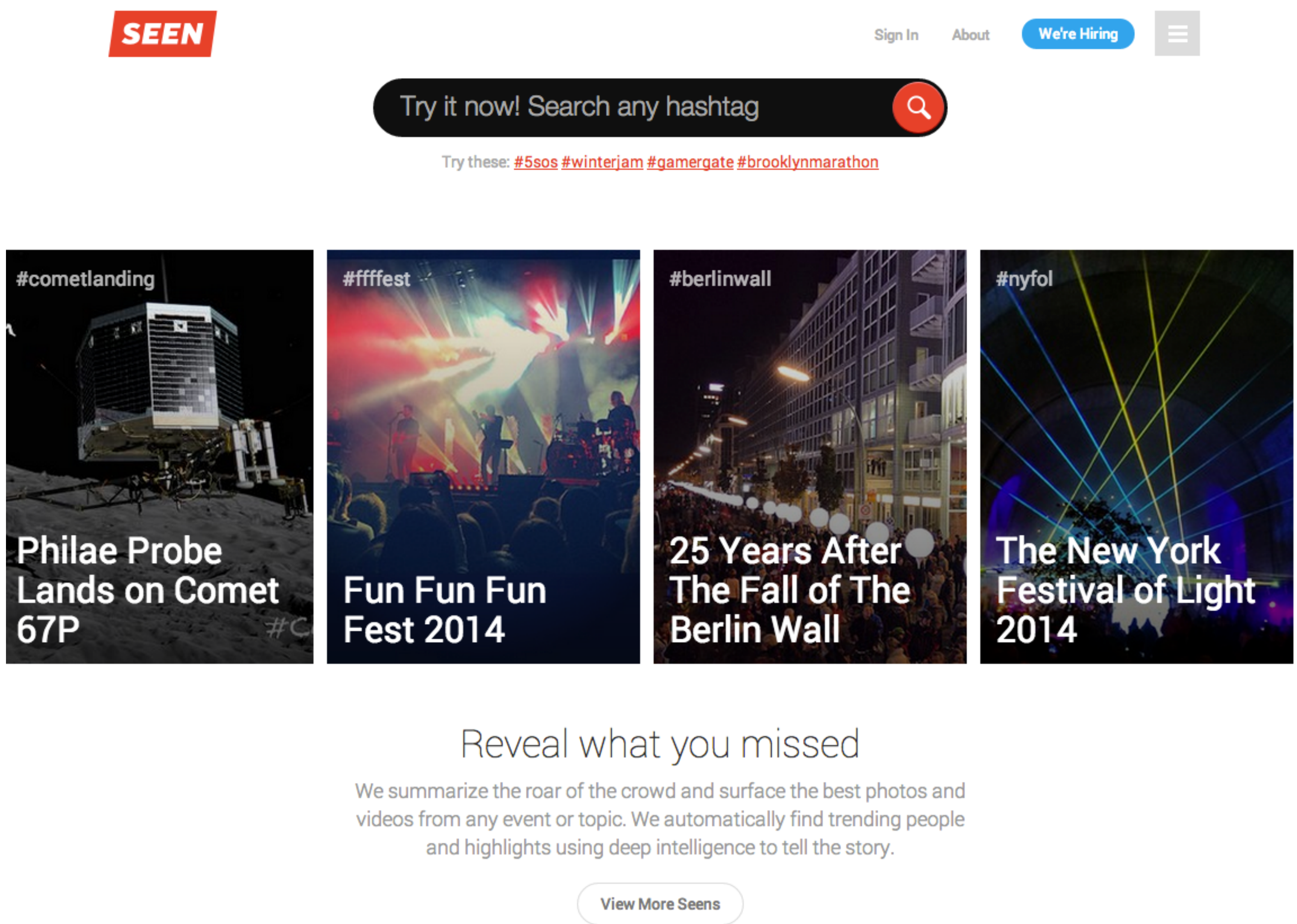


Brutus | Mark Anthony | Cleopatra

Events in Social Media Streams

- **WWW2015** conference will be held in **Florence.**
- **Hofmann** confirmed keynote at **WWW2015** in **Florence**
- **WWW2015** opens **May 20** in **Florence**

Event Detection



Dynamic Community Detection Algorithms

Most algorithms assume a **single static graph** in input.

Naive solution: run the algorithm **once** for **each update**.

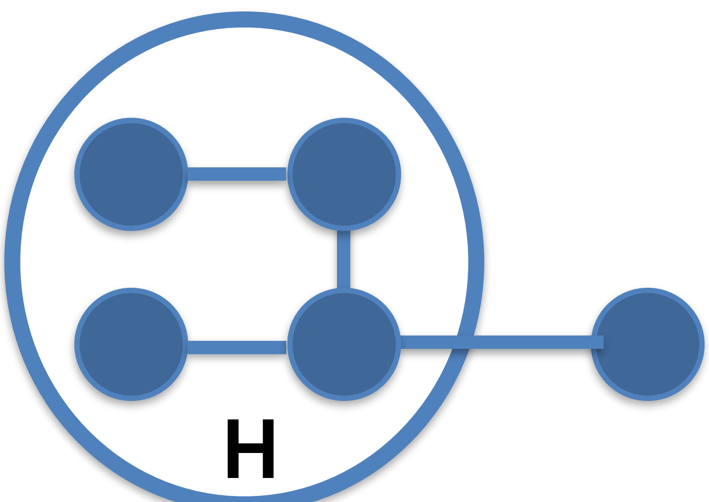
GOAL: efficiently keep track of the communities as the graph evolve.

Densest Subgraph

Definition (Densest Subgraph Problem)

Let $G = (V_G, E_G)$ be an undirected graph. Find a subgraph $H = (V_H, E_H)$ of G with maximum *average degree density*:

$$\rho(H) = \frac{|E_H|}{|V_H|}.$$



Density H = 3/4

Densest Subgraph

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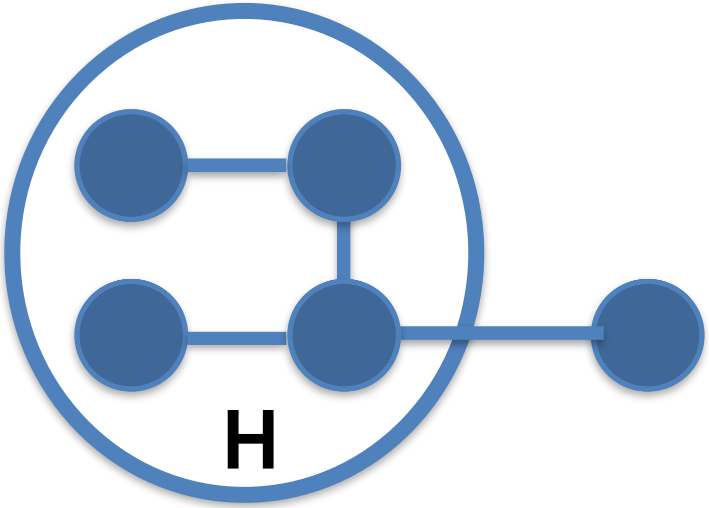


Densest Subgraph in Static Graphs

- **Community** used in Social Networks, Web and Biology.
- **Polynomial** exact algorithm (Goldberg, 1984)
- **(2+eps)-approximation** MapReduce algorithm (Bahmani et al., 2012).

Densest Subgraph in Dynamic Graphs

No results known* in dynamic graphs with **sublinear update time** (*before our publication*).

Naive Approach: O(m + n) time per update!

* **Bhattacharya et al**. - to appear in STOC 2015. Strong guarantees in streaming model.

Our Problem

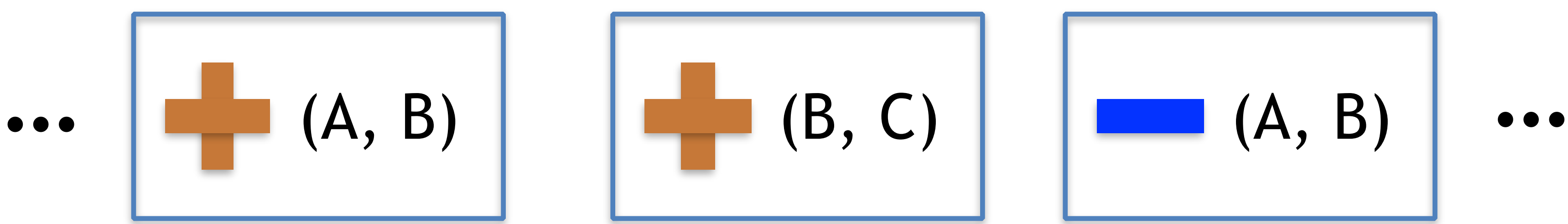
Goal: Preserve a **2+eps** approximation with average time **O(poly-log(n+m))** per update.

Notice: Much better than **O(n+m)** per **update** and **includes output time**!

Our Dynamic Graph Model

Start from an **empty graph**.

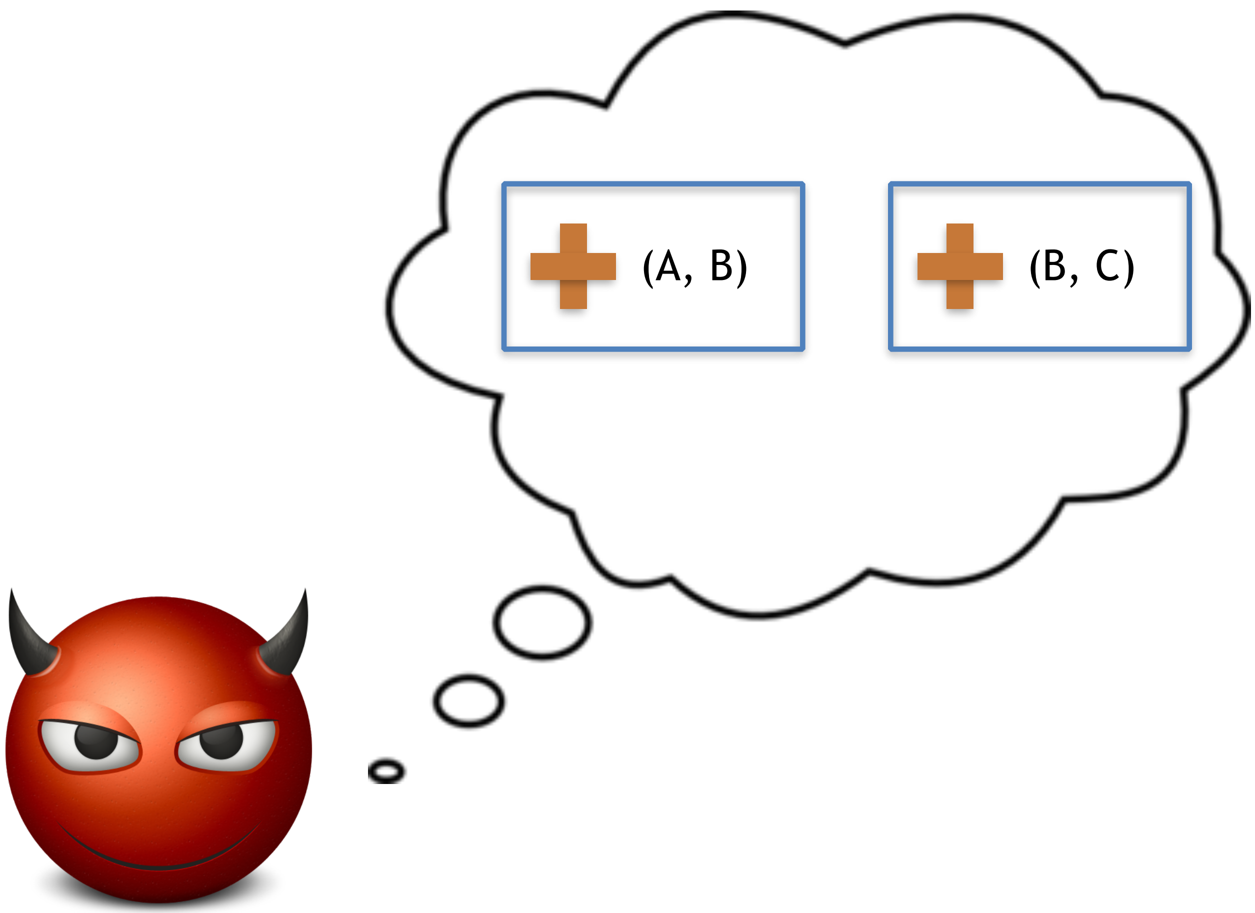
Arbitrary long **sequence of edge** updates arrives…



This models also **node addition/removals** implicitly.

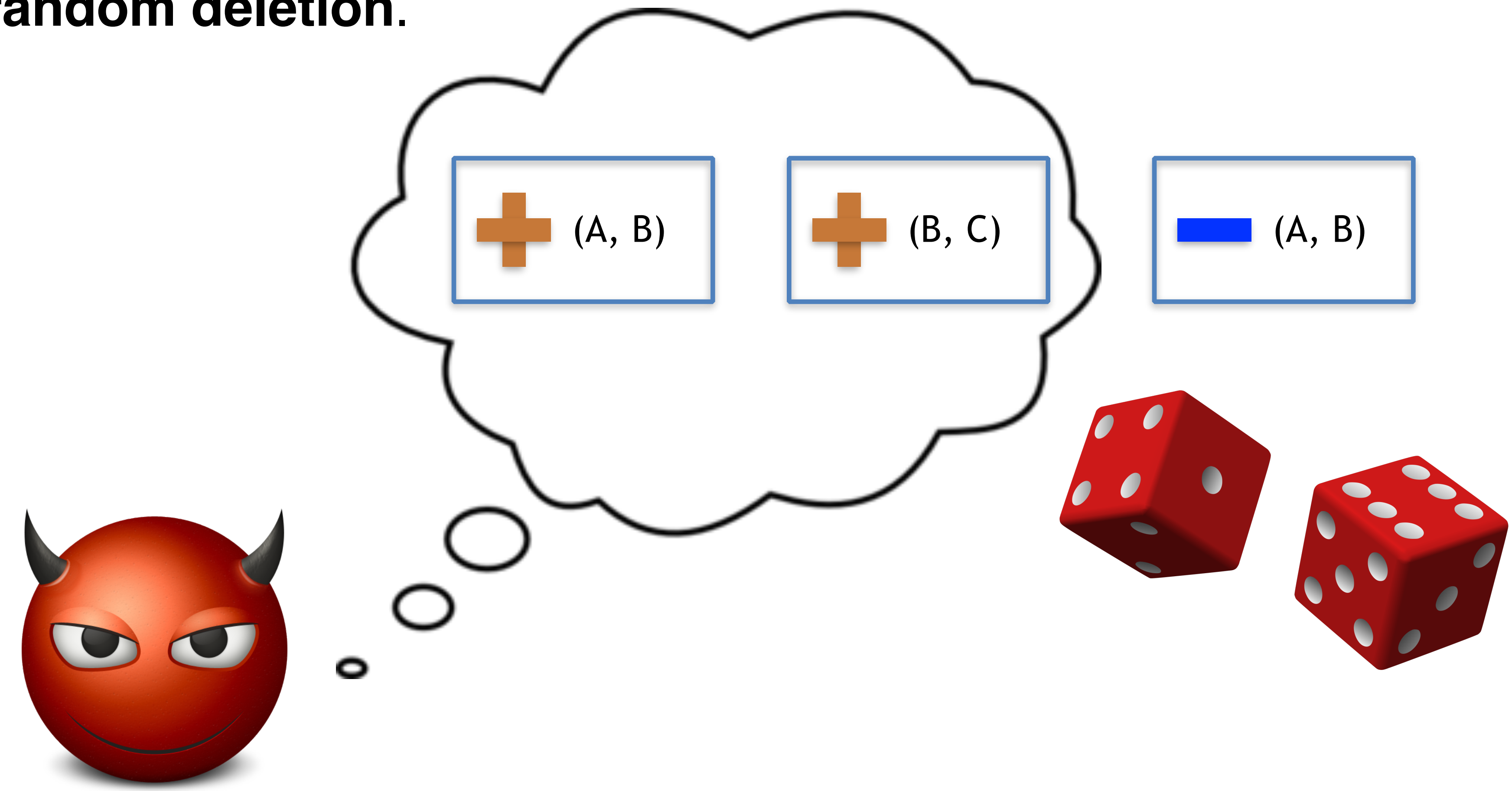
Incremental and Fully-Dynamic

INCREMENTAL: **arbitrary** stream of edges **additions only**.



Incremental and Fully-Dynamic

<u>FULLY-DYNAMIC:</u> stream of edges **arbitrary additions** and **random deletion**.



Our Goal

Design a Data Structure:

1) **AddEdge(u,v)**
2) **RemoveEdge(u,v)**

Both operations **can output** a **new** densest subgraph **S** or **nothing.**

Invariant: the **last subgraph in output** is a **2+eps** approx. for the **current graph**

Result for edge additions (incremental)

Theorem: We maintain a **2+eps** approx. in **O(log^2(n) / eps^2)** average time and **linear space**

Significant improvement over **naive approach: O(m+n) average time**

Result for edge additions and deletion (fully dynamic)

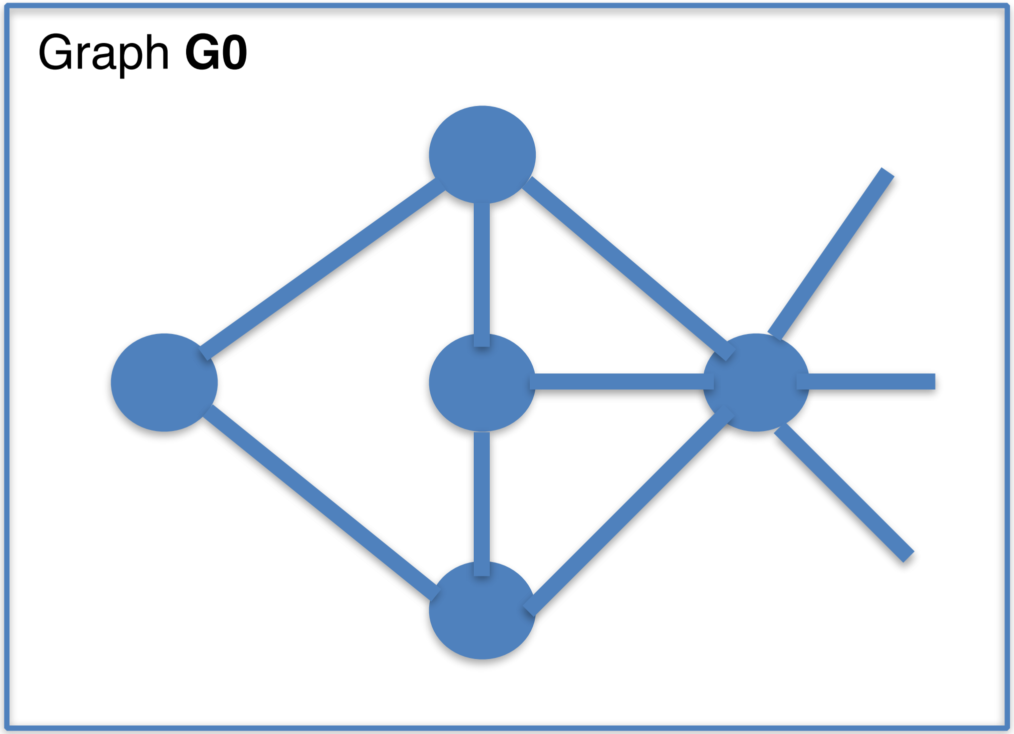
Theorem: We maintain a **2+eps** approx. in **O(log^4(n) / eps^4)** average time and linear space.

Very fast **also in practice!**

Roadmap

- Review **Bahmani et al.** for static graphs.
- A **new static graph** algorithm.
- **Incremental** algorithm.
- Randomized **fully-dynamic** algorithm.

Static Case - Bahmani et al. Algorithm

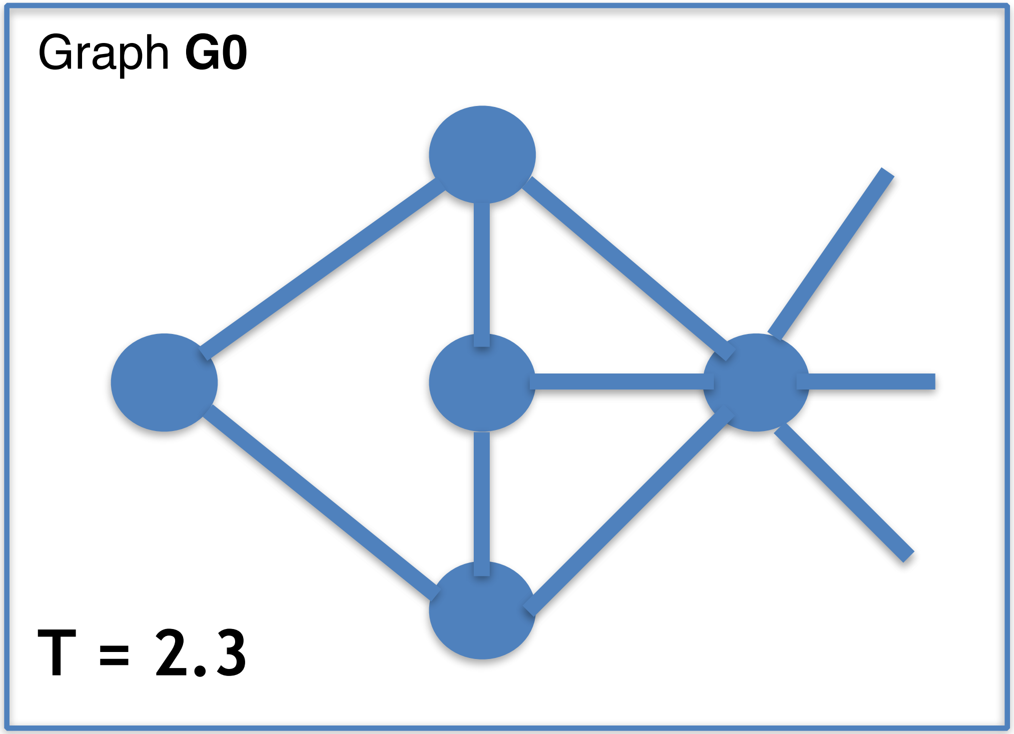


Let eps > 0:

Iteration: 1

1) Compute Avg. Deg = K

Static Case - Bahmani et al. Algorithm



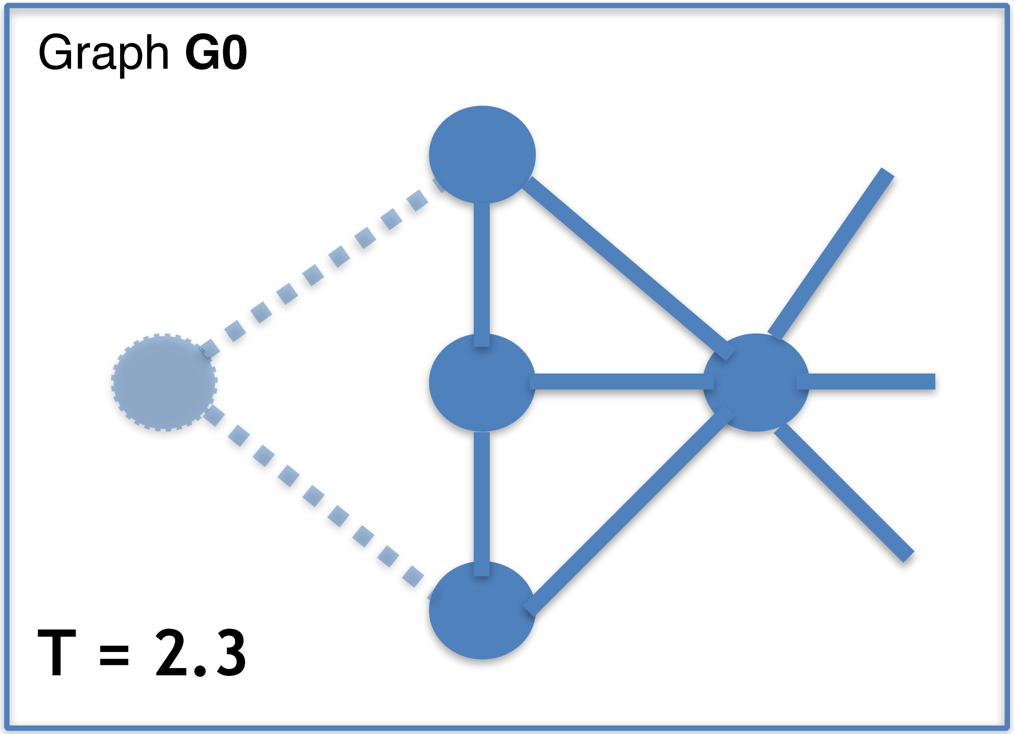
Let eps > 0:

Iteration: 1

1) Compute Avg. Deg = K

2) Let T = K (1+eps)

Static Case - Bahmani et al. Algorithm



Let eps > 0:

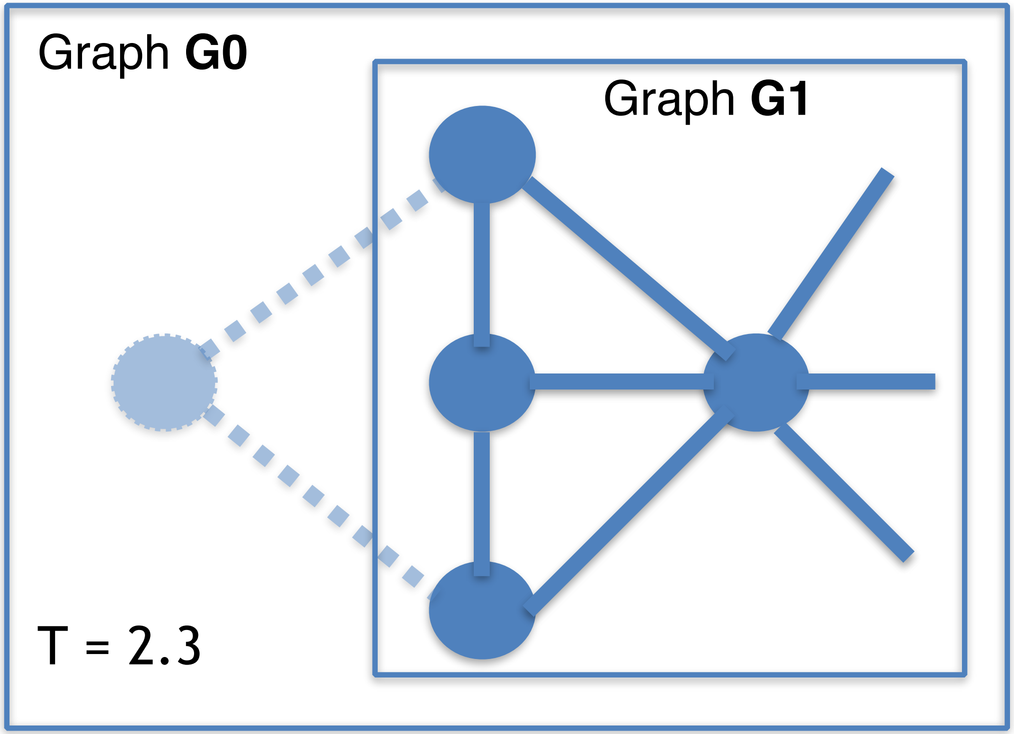
Iteration: 1

1) Compute Avg. Deg = K

2) Let T = K (1+eps)

3) Remove nodes with degree < T

Static Case - Bahmani et al. Algorithm

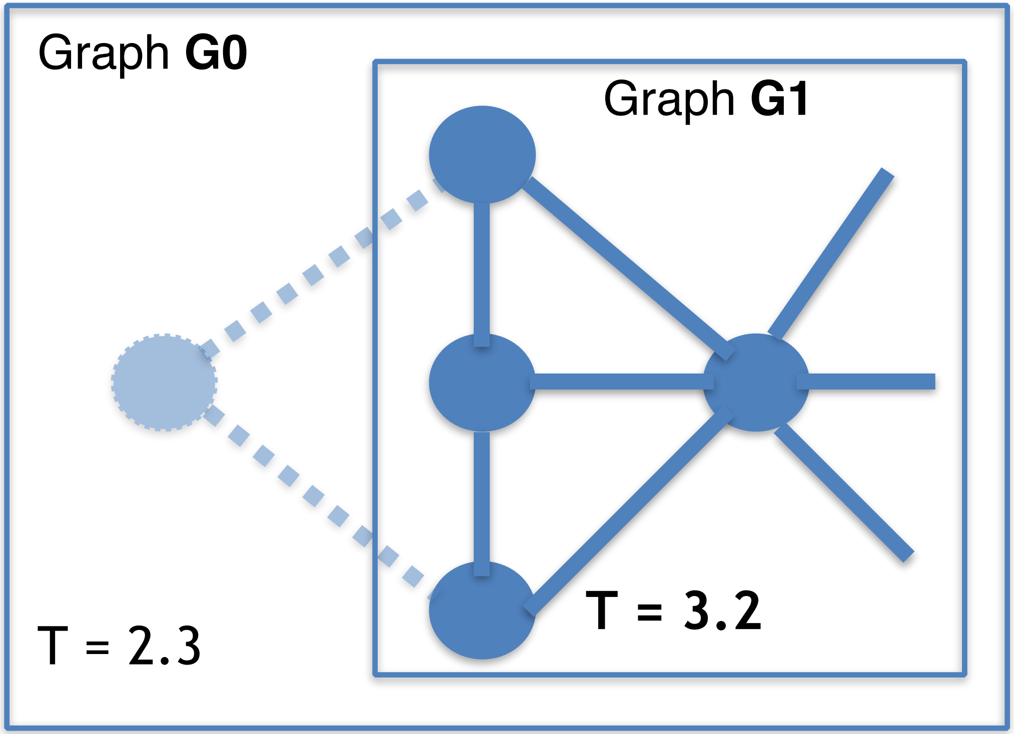


Let eps > 0:

Iteration: 2

1) Compute Avg. Deg = K

Static Case - Bahmani et al. Algorithm



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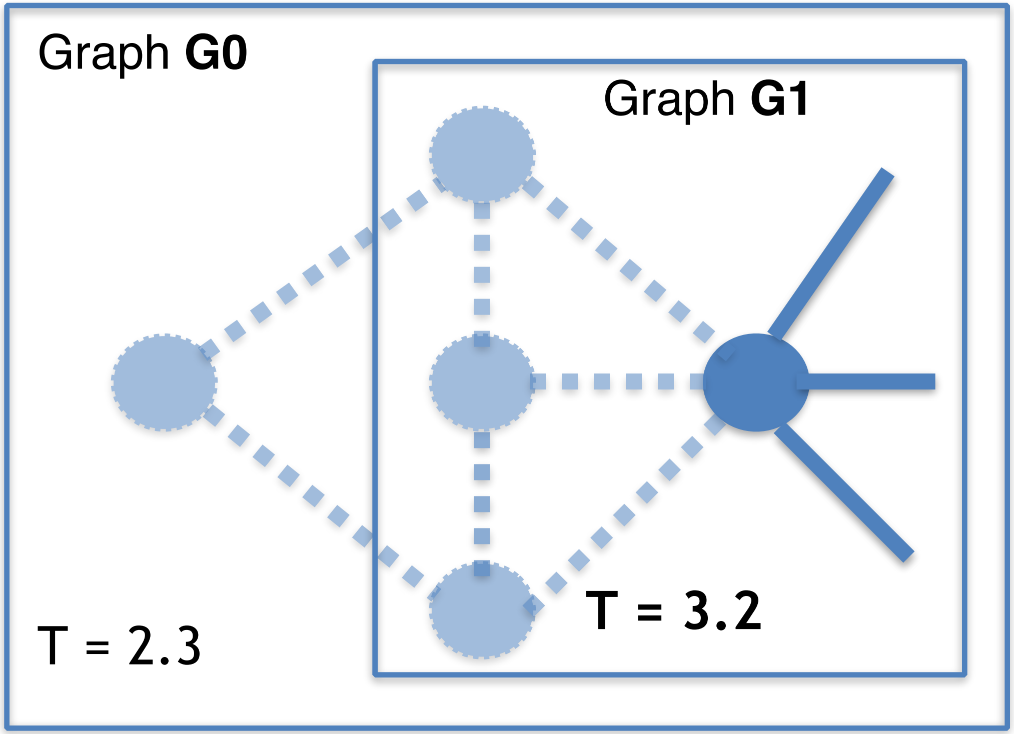

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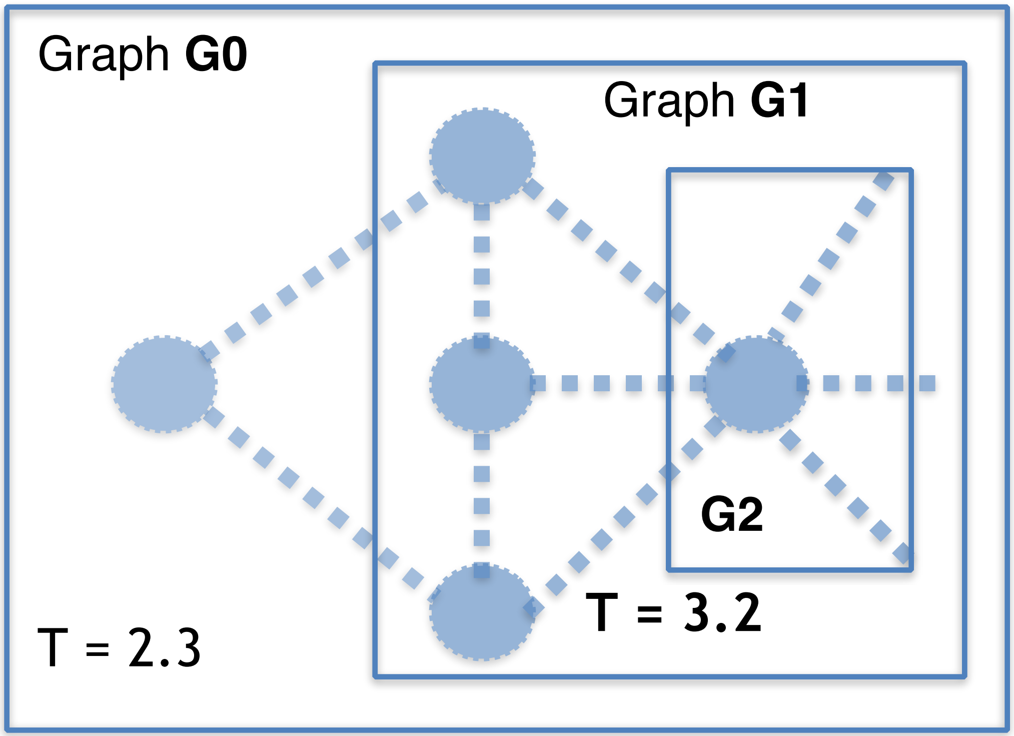
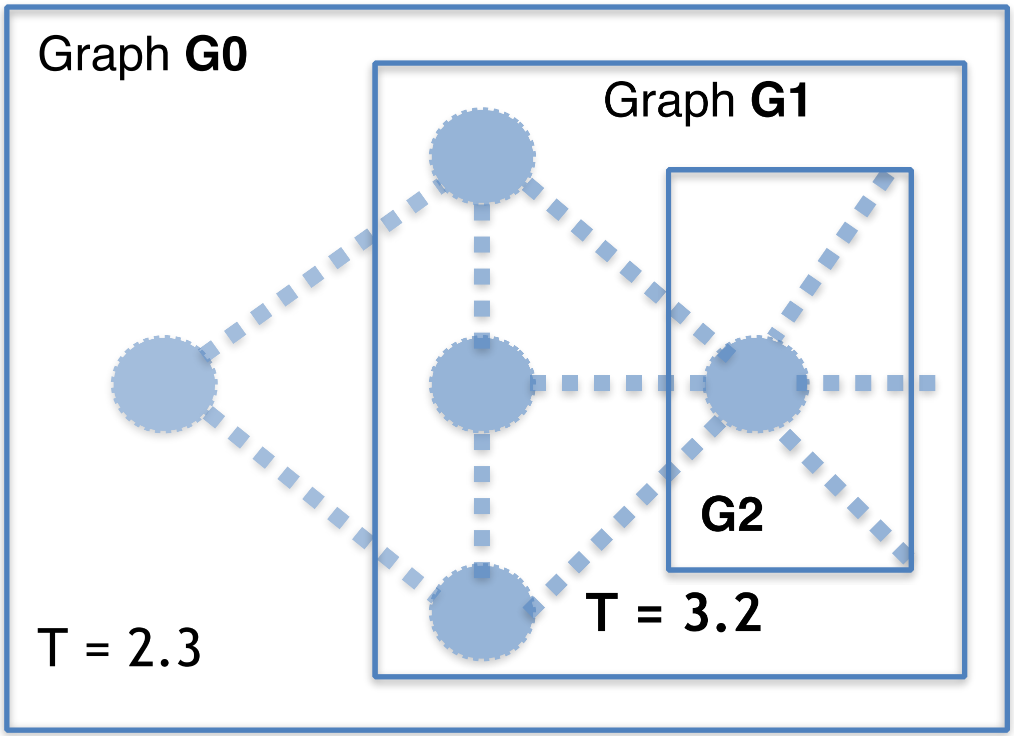


Iterate until **all nodes are removed.**

Output the densest subgraph **Gi.**

Static Case - Bahmani et al. Algorithm



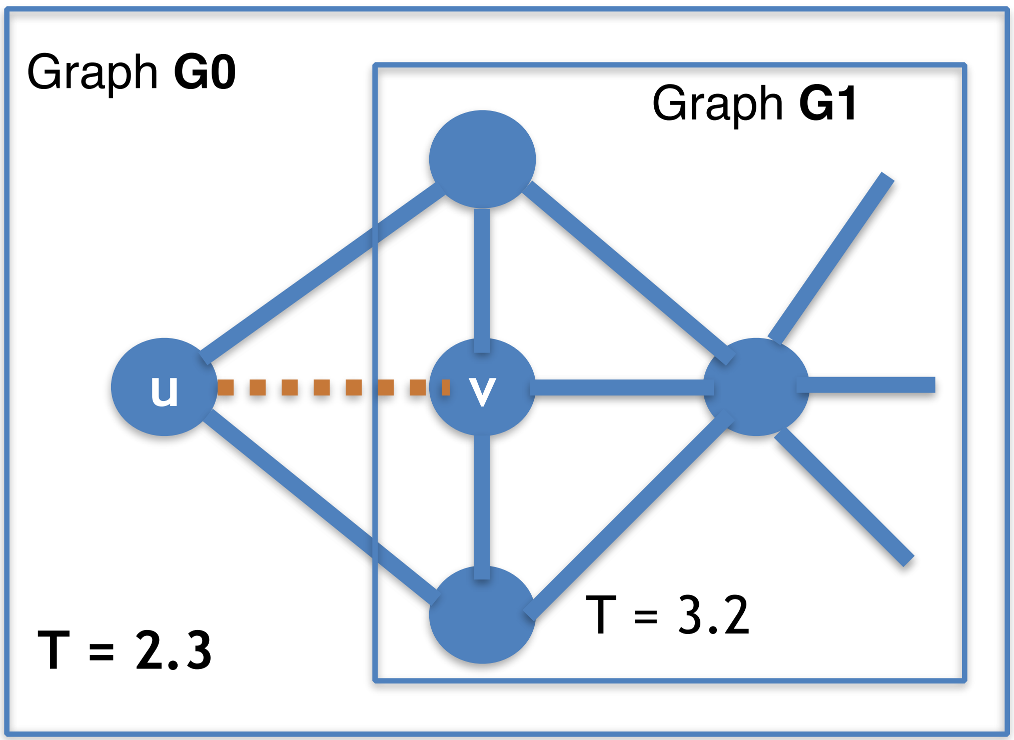
Iterate until **all nodes are removed.**

Output the densest subgraph **Gi.**

Theorem: (Bahmani et al.)
2+eps approx. in **log(n) steps**.

Towards a Dynamic Algorithm

- **Idea:** Store graphs **Gi**'s.
- When an edge is **added** update the **Gi's**



This ensures a 2+eps approximation!

Towards a Dynamic Algorithm

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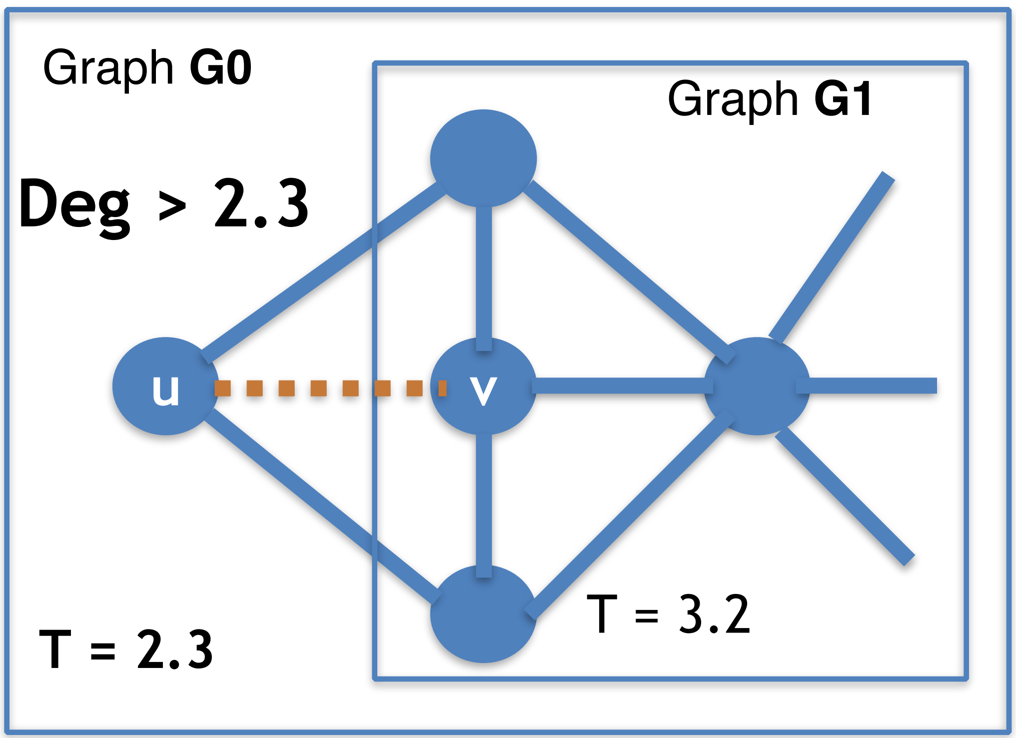

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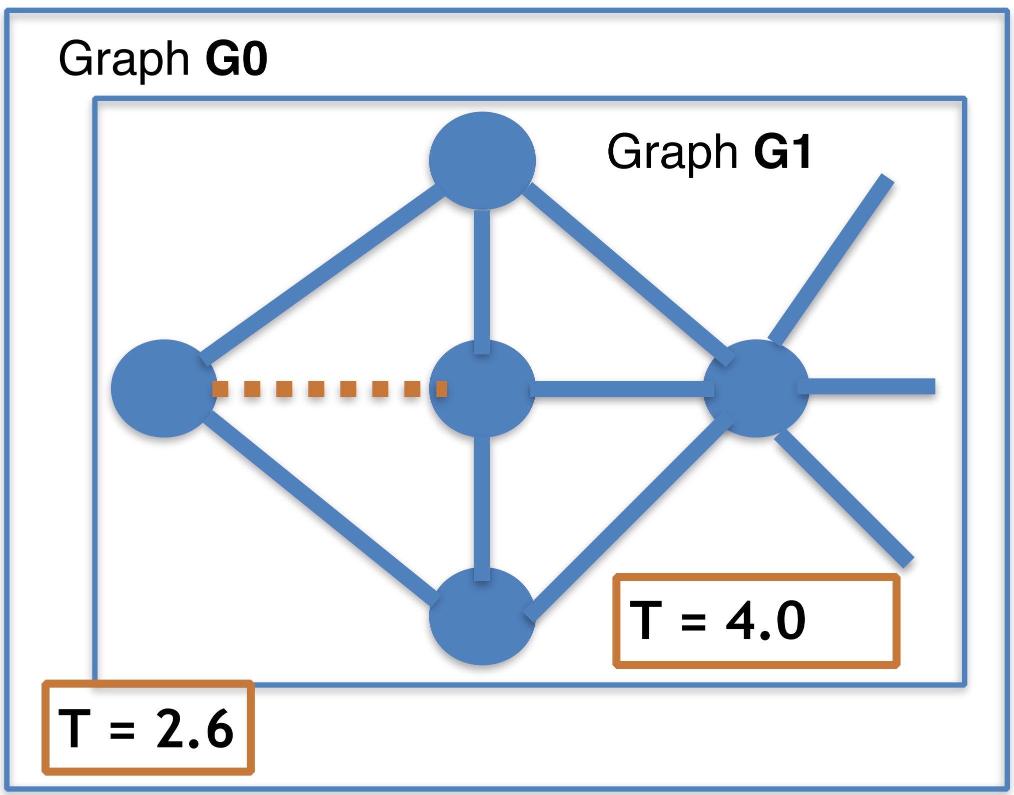
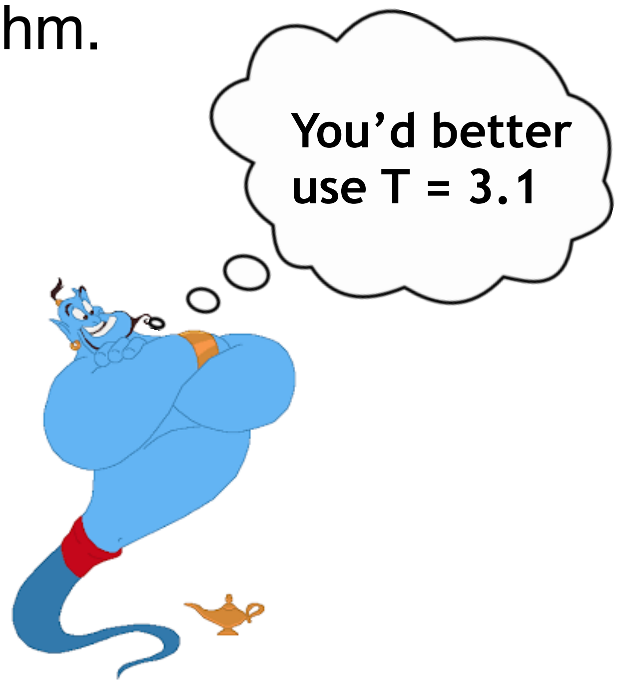


Chain effect!

Idea: fix Threshold T for all iterations

- Use same threshold T at each iteration.
- **Easier** to **analyze** and **maintain.**

For correct threshold **T**: same approximation of Bahamani et al.'s algorithm.



Moving Threshold (Only Additions)

1) Set T = 1 to compute densest subgraph **H** and output it.

This provides a 2+eps approx.
in **O(poly-log(n)) average time**

Moving Threshold (Only Additions)

1) Set T = 1 to compute densest subgraph **H** and output it.

2) Maintain the **Gi'** using threshold **T as long as all nodes** are removed in **O(log(n))** steps.

This provides a 2+eps approx. in **O(poly-log(n)) average time**

Moving Threshold (Only Additions)

1) **Set T = 1** to compute densest subgraph **H** and output it.

2) Maintain the **Gi'** using threshold **T as long as all nodes** are removed in **O(log(n))** steps.

3) **Repeat** from **1)** with higher threshold **T = T * 2**

This provides a 2+eps approx.
in **O(poly-log(n)) average time**

Fully-Dynamic Case

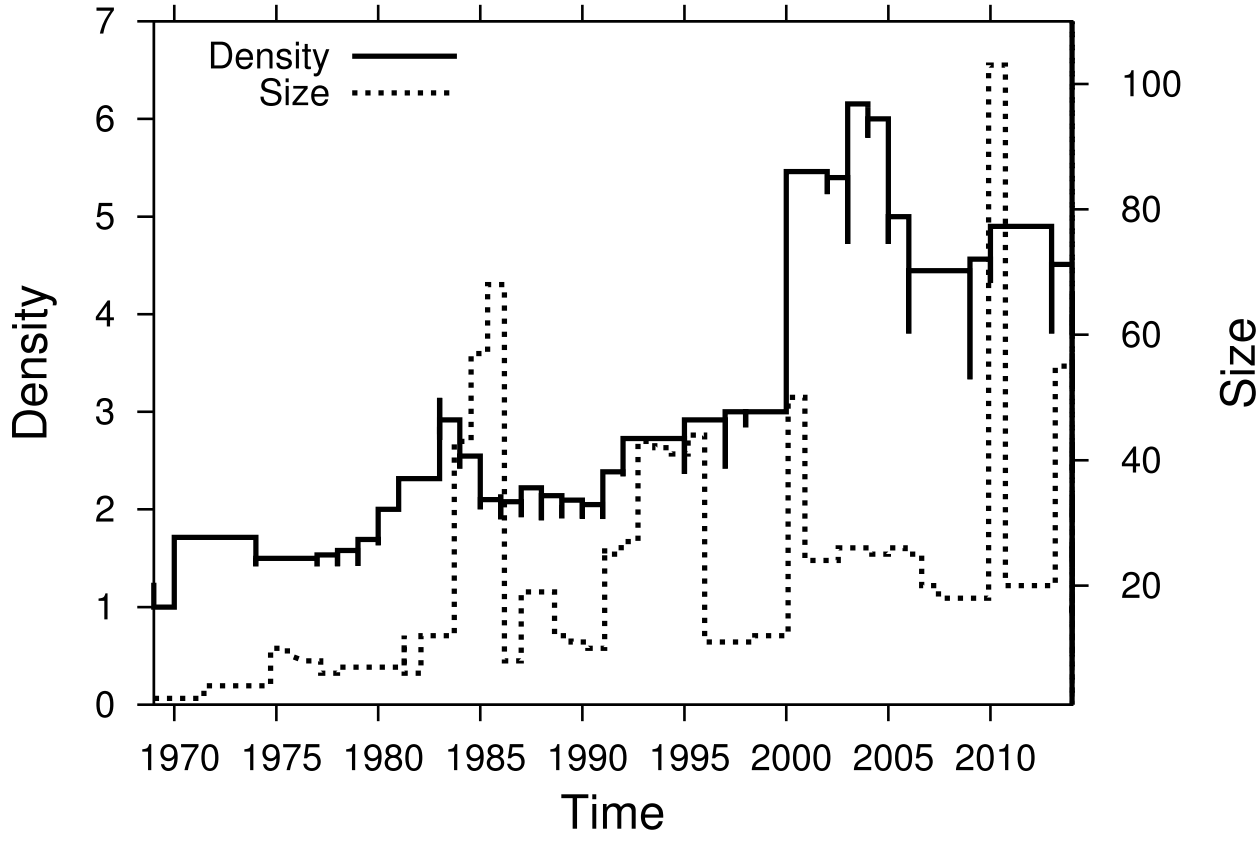
The analysis is significantly harder:

- The density can **increase/decrease** in complex patterns…
- …densest subgraph is **stable** under **random removals.**
- **We tackle the stability** to recompute the subgraph few times.

Experimental Evaluation - Datasets

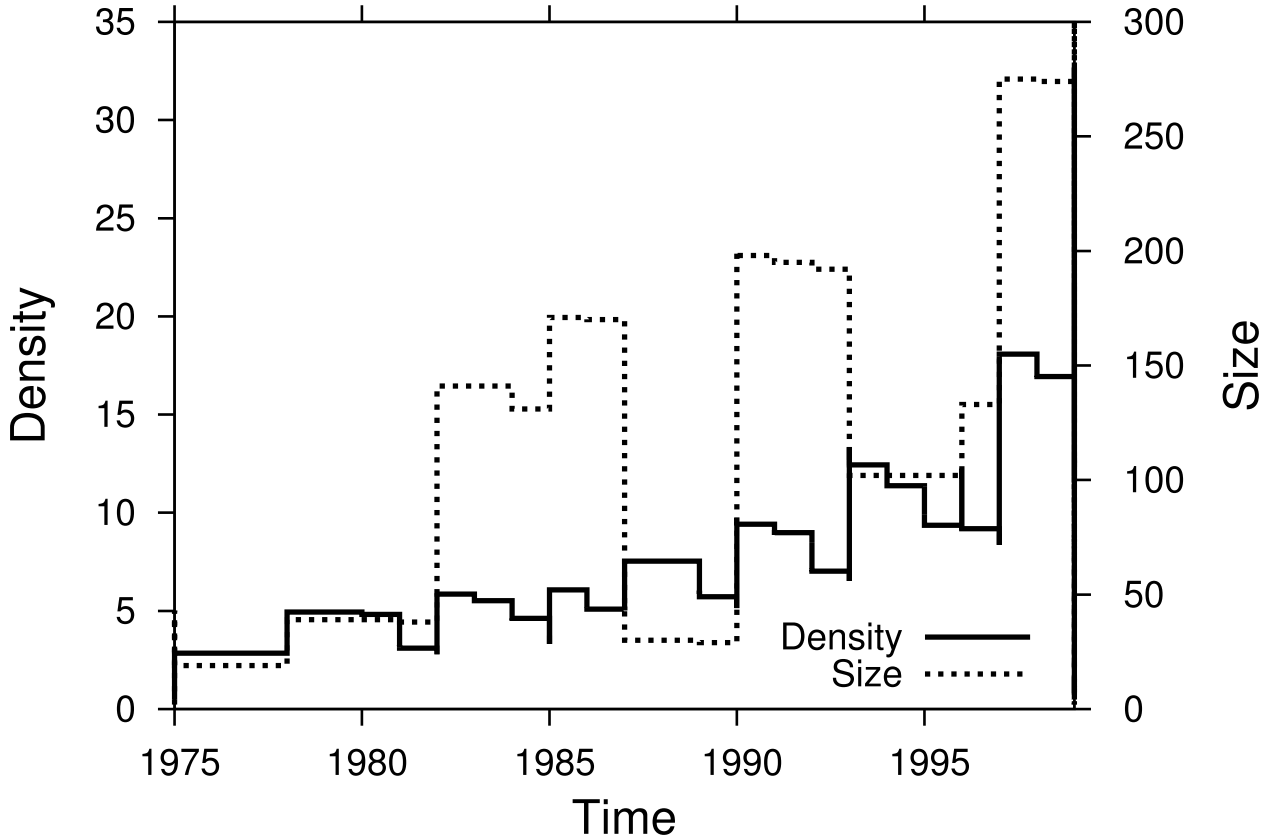
- **DBLP& Patent:** co-authorship graph.
- **LastFM:** songs co-listened.
- **Yahoo! Answers: >1 Billions** edges. Edge if two users answer the **same question.**

Evolution Densest Subgraph



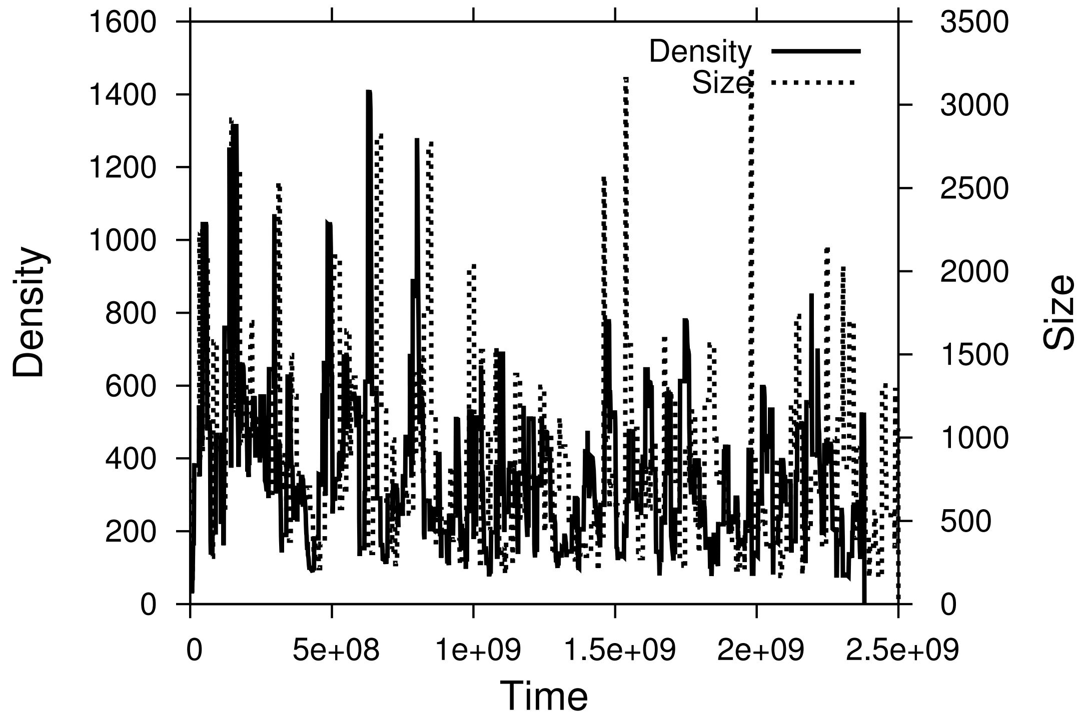
DBLP - Sliding Window 5 years

Evolution Densest Subgraph



Patent Citations - Sliding Window 5 years

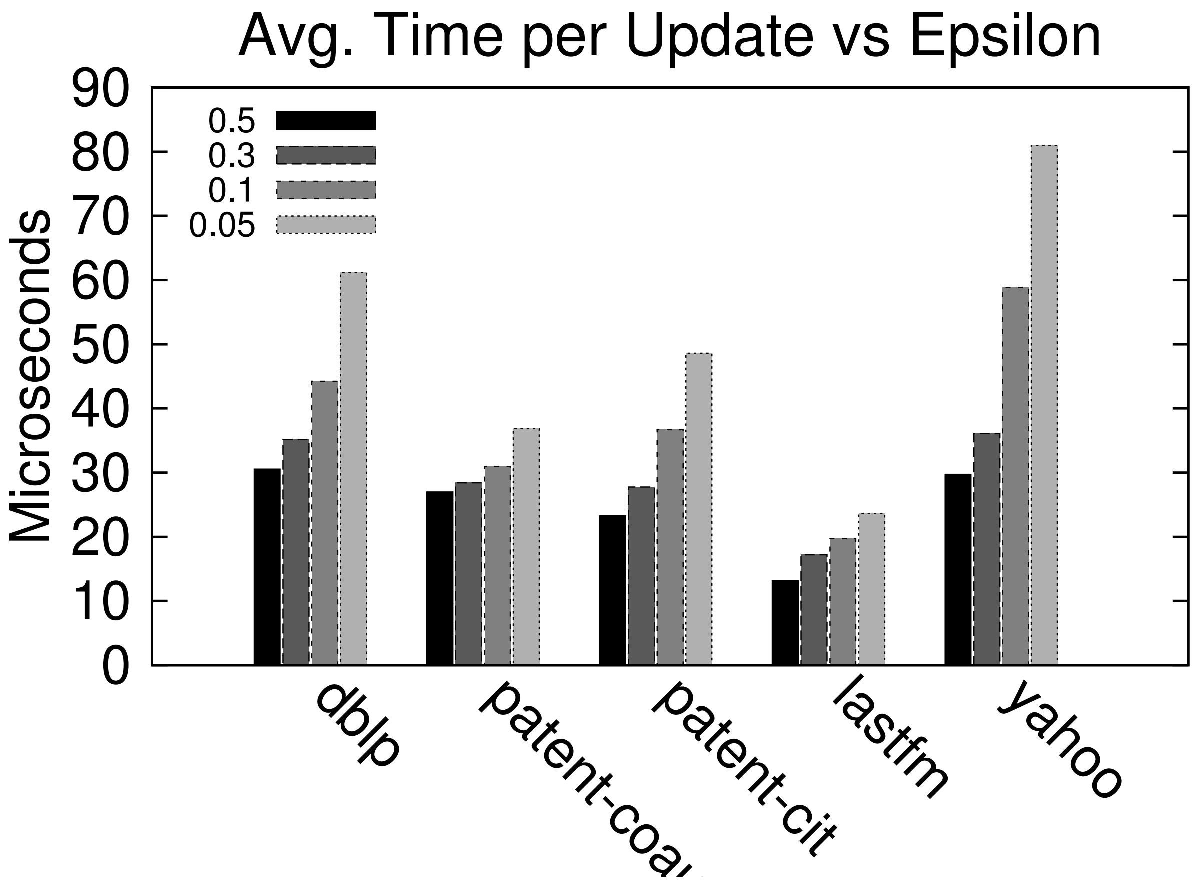
Evolution Densest Subgraph



Efficient in **Highly Dynamic** Datasets with **Billions** of Updates.

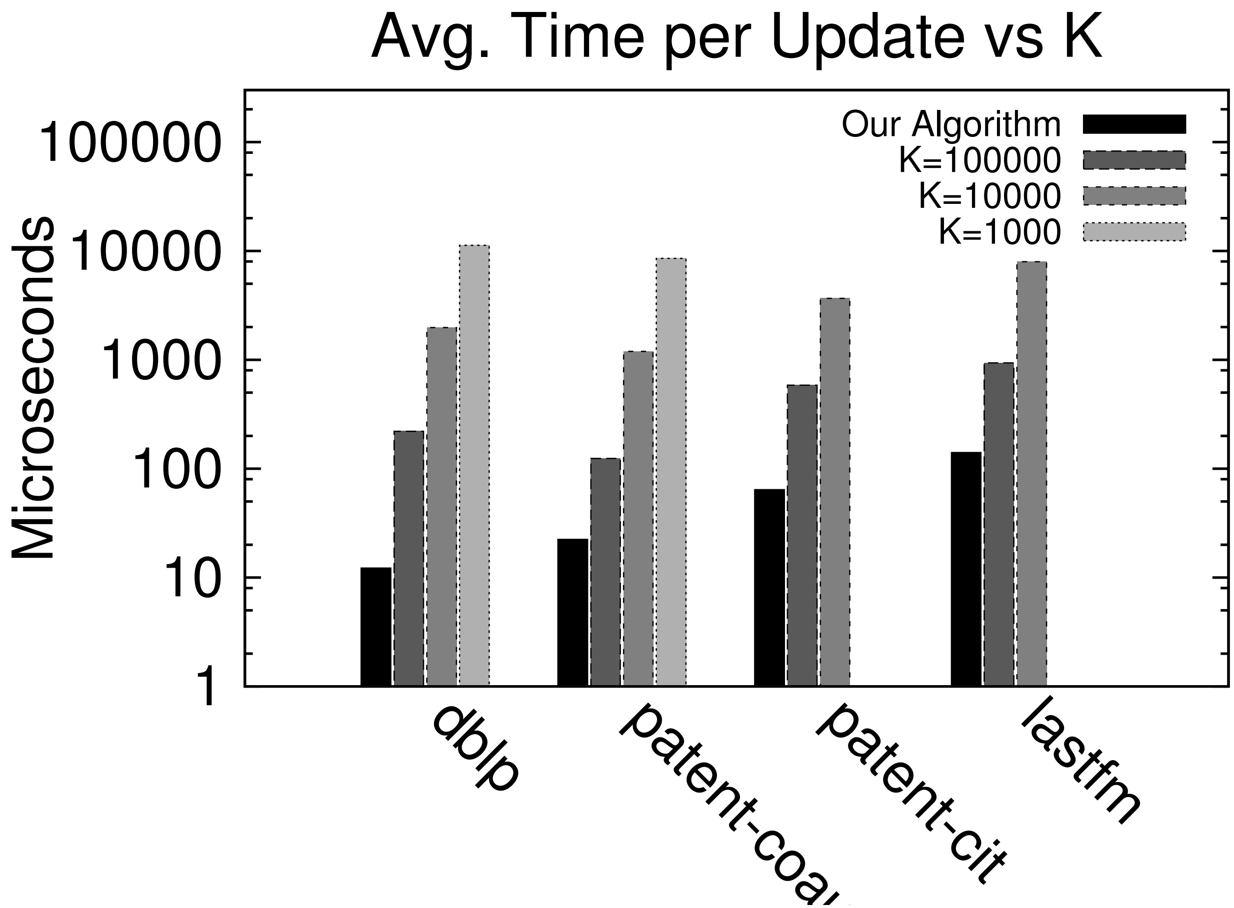
Yahoo Answers - Sliding Window 100M edges

Update Time vs Epsilon

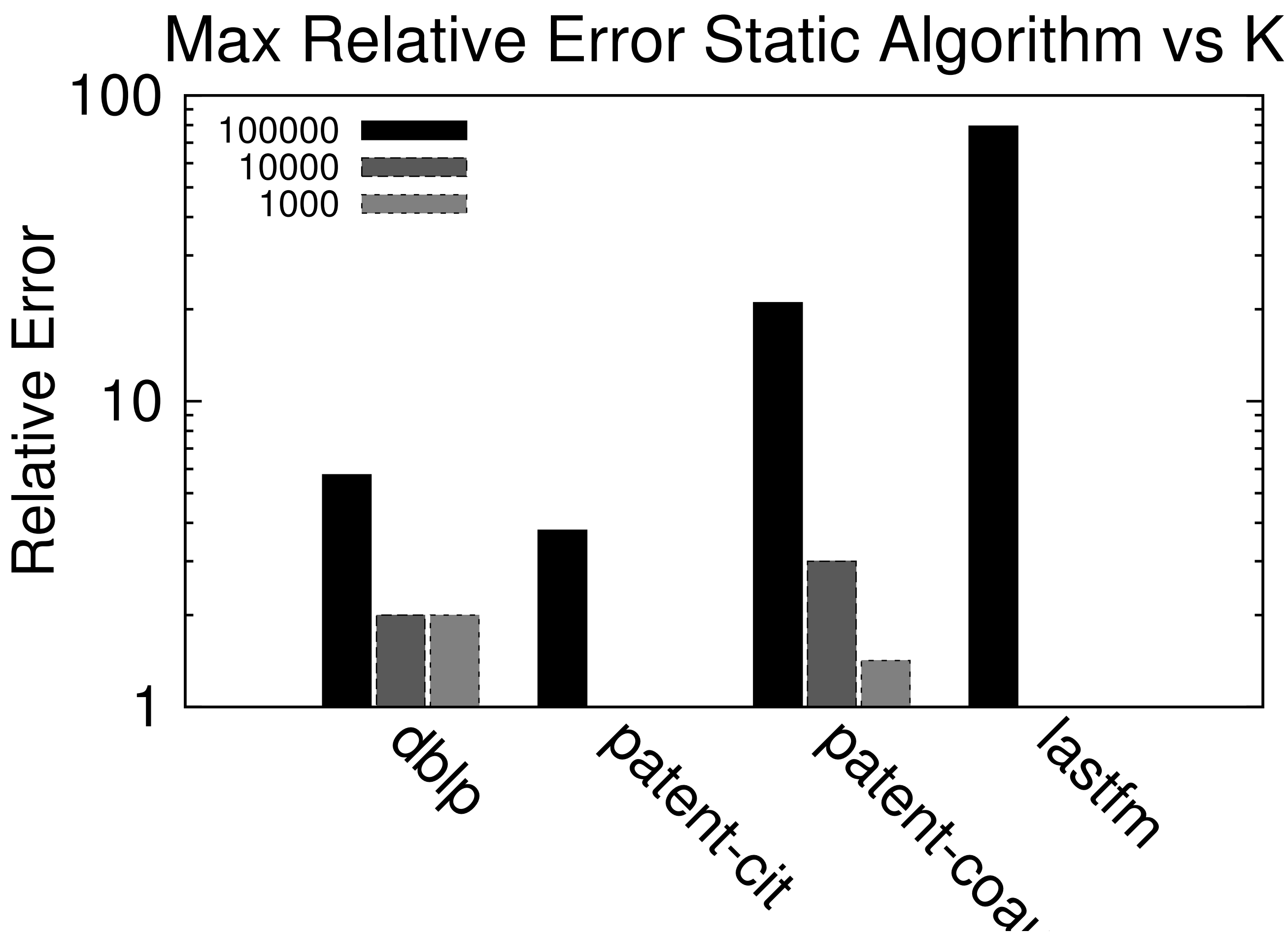


Scales much better with Epsilon than worst case.

Comparison with Static Algorithm



Comparison With Static Algorithm



Conclusions and Future Work

- It is possible to maintain the densest subgraph efficiently in dynamic graphs.
- **Future work:** Recent Techniques (**Bhattacharya et al.)** to define 2+eps with adversarial removes?
- Top-k Densest Subgraph in Dynamic Graphs.

Thank you for your attention

Recent Results - STOC

Concurrently to our work **Bhattacharya et al., STOC 2015** introduced a novel streaming algorithm for densest subgraph with **strong guarantees**.

- Different model: Update vs Query time.
- Strong space constraints (cannot store entire graph).
- Adversarial additions and deletions.

- 4+eps approx with O(n poly log) space, O(poly log) update time, O(n) query time.
- 2+eps approx with O(n poly log) space, higher time complexity.

Incremental Case: Only Additions

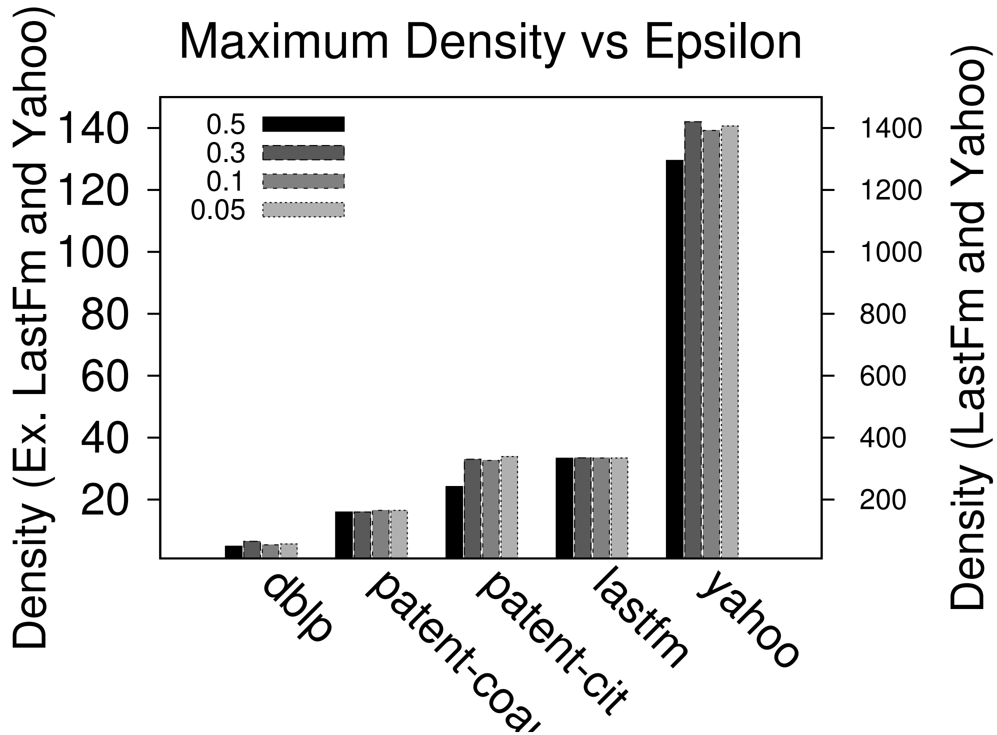
Lemma

During each round we can maintain the invariant with total cost $O(m\log(n)\epsilon^{-1})$.

Lemma

For any sequence of m edges additions, there are at most $O(\log(n)\epsilon^{-1})$ *rounds in total.*

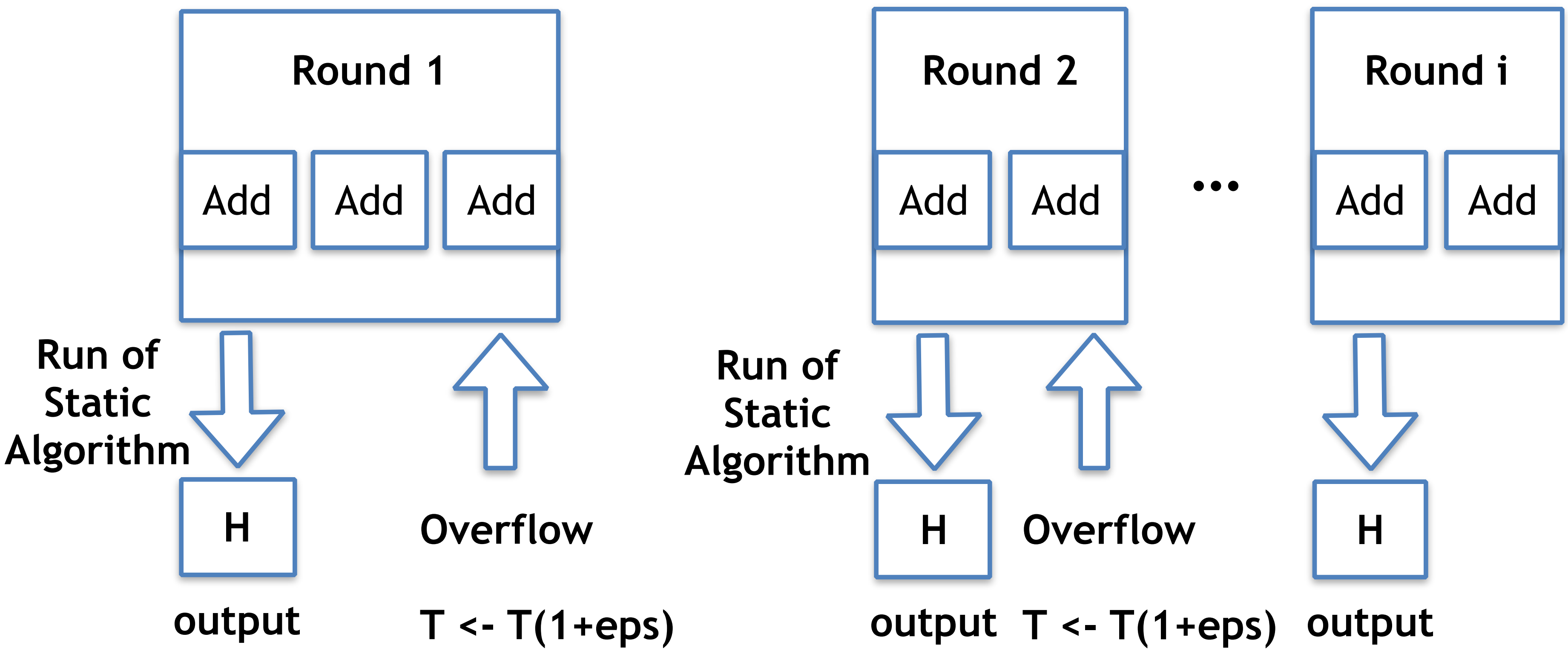
Density vs Epsilon



Max density is **stable** with different epsilons.

Analysis of the Algorithm

We divide the edge additions in **Rounds.**



Densest Subgraph - LP Primal

There is a $[0,1]$ variable y_i for each node $v_i \in V$, while there is a $[0,1]$ variable x_{ij} for each edge $e_{ij} \in E$.

$$
\begin{aligned}
\max \quad & \sum_{ij \in E} x_{ij} & & \text{(LP Primal)} \\
\text{s.t.} \quad & x_{ij} \le y_i & \forall e_{ij} \in E, \quad & (5) \\
& x_{ij} \le y_j & \forall e_{ij} \in E, \quad & (6) \\
& \sum_{i \in V} y_i = 1, & & (7) \\
& x_{ij}, y_i \in [0,1] & \forall e_{ij} \in E, \forall v_i \in V. \quad & (8)
\end{aligned}
$$

Definitions

We say that an algorithm is **a approximation** of the densest subgraph problem for **a > 1** if it outputs a graph with density at least:

OPT / a

We say that an operation has **T amortized time** if for any sequence of **k** update operations the total time is

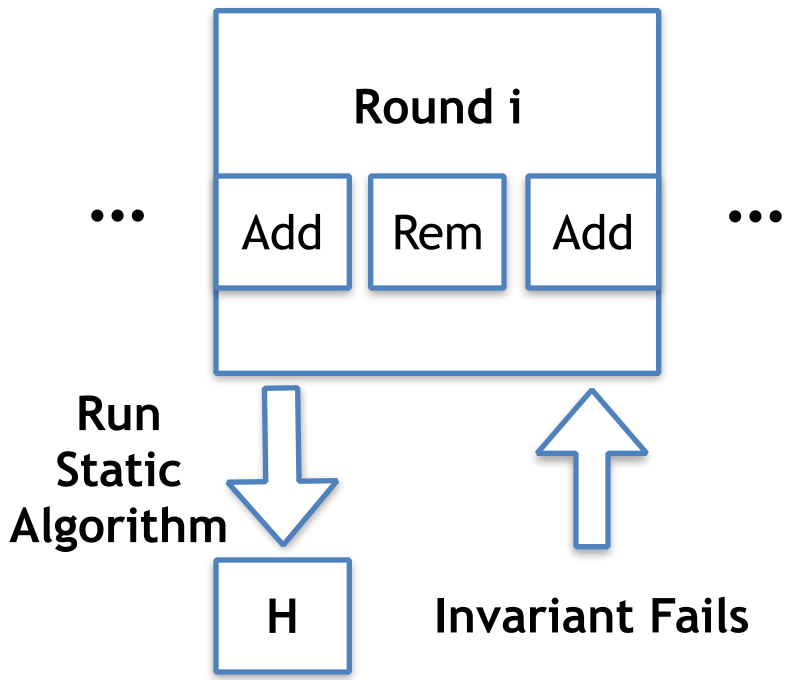
O(k T)

Densest Subgraph - LP Primal Dual

- The dual problem is the well-known graph orientation problem.
- Given undirected graph G find directed graph H obtained orienting the edges of G arbitrarily, that **minimizes** the **maximum** in-degree.
- If G has orientation of max in-degree $< D$ then density of densest subgraph is $< D$.
- Hence, if it is possible to remove all nodes by recursively removing nodes with degree $< D$ then max density is $< D$.

Fully Dynamic Algorithm

We divide the edge additions and deletions in **Rounds.**



Fully Dynamic Algorithm

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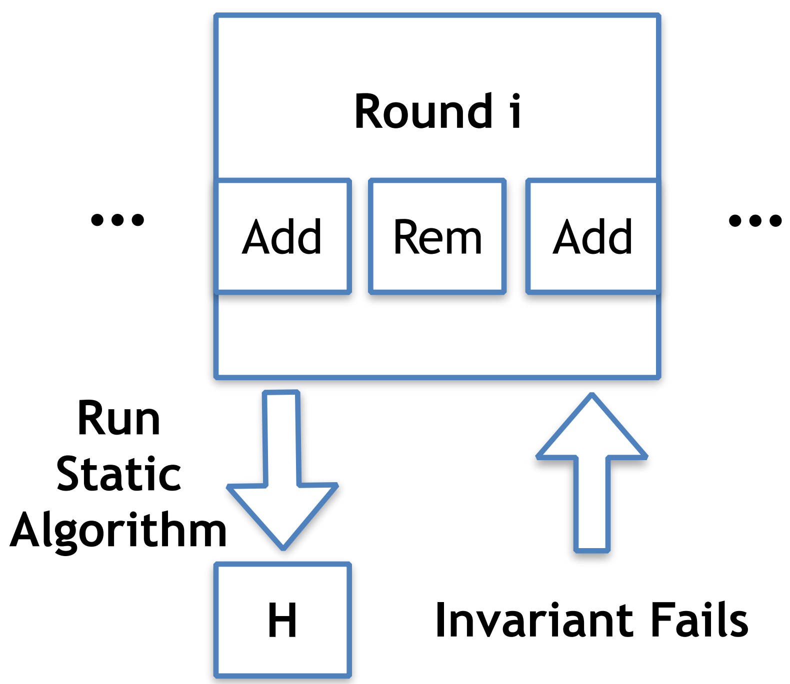
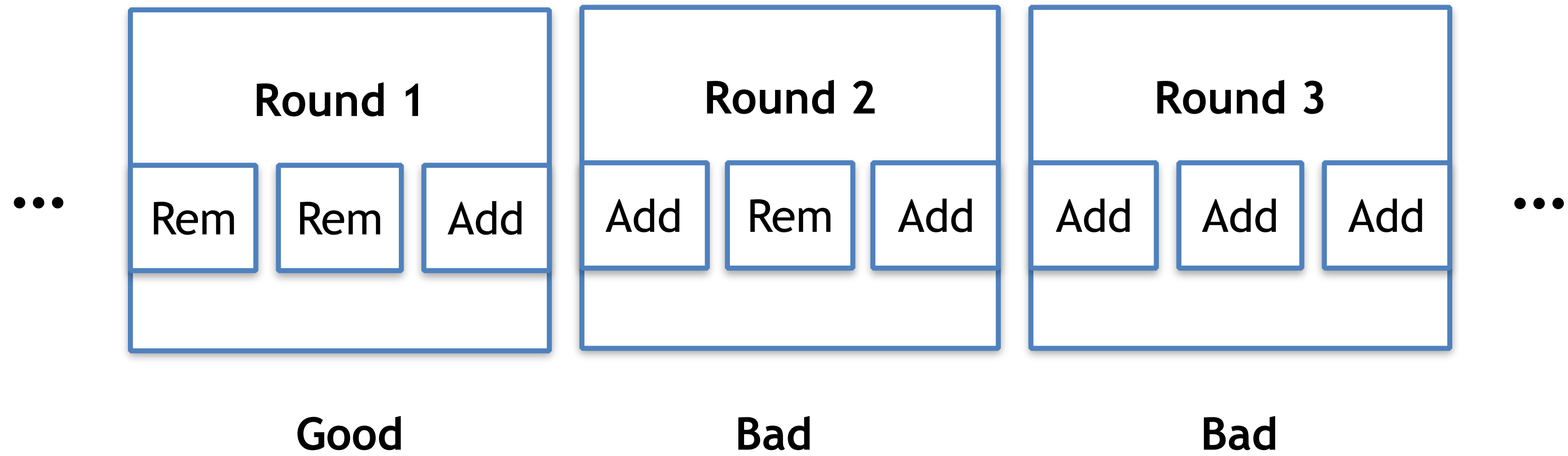


Bad Round < O(m / log(n)) removals

Good Round > O(m / log(n)) removals

Fully Dynamic Algorithm



Idea: in good rounds **removals "pay"** for all the operations

We can show that there are never more than poly-log **consecutive bad rounds** (w.h.p)